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Line Insertions in Totally Positive Matrices

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It is obvious that between any two rows (columns) of an m-by-n totally nonnegative matrix a new row (column) may be inserted to form an $(m+1)$-by-n (m-by-$(n+1)$) totally nonnegative matrix. The analogous question, in which "totally nonnegative" is replaced by "totally positive" arises, for example, in completion problems and in extension of collocation matrices, and its answer is not obvious. Here, the totally positive case is answered affirmatively, and in the process an analysis of totally positive linear systems, that may be of independent interest, is used. © 2000 Academic Press

I. INTRODUCTION

An m-by-n matrix A is called *totally positive* (*nonnegative*) if *every* minor of A is positive (nonnegative). See [A, GK, GM, K] for background and ample motivation. The following interpolation question often arises in totally positive (nonnegative) completion problems, and it seems not to have been addressed in the literature on the subject nor is its answer known to others working in the field. Under what circumstances may an

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additional row (column) be inserted into an m-by-n totally positive (nonnegative) matrix A, say between row (column) i and $i+1$ (j and $j+1$), $i=0, ..., m$ ($j=0, ..., n$), so that the resulting $(m+1)$-by-n (m-by-$(n+1)$) matrix is totally positive (nonnegative)? To combine the two cases, call a row or column a *line*, and refer to this as the problem of line insertion.

The following observations about line insertions may be easily made.

(i) In either the totally positive (TP) or totally nonnegative (TN) case, the set of possible insertions (in a specific place) is convex, because of linearity of the determinant as a function of a line.

(ii) In the TN case, the set of possible insertions is always nonempty (i.e., line insertion is always possible in every position in every TN matrix) because either repetition of an adjacent line or use of the zero line suffices. However, none of these insertions is possible in the general TP case, and the existence of a solution to the line insertion problem in the TP case, though plausible, is not obvious.

(iii) When the line is exterior (top/bottom, right/left), it is obvious that there is a solution in the TP case by adding sufficiently large entries one-at-a-time in the correct order. For example, to add a new last column, start from the top and move down the column; each successive entry enters only positively into every minor it completes, so that a sufficiently large value suffices each time.

We show here that the life insertion problem always has a solution in the TP case and the mechanism may be of independent interest (as we have found it to be in other contexts). We give a careful analysis to "totally positive linear systems." Throughout we let det A denote the determinant of A.

II. MAIN RESULTS

The following lemma is fundamental to the insertion strategy and may be of independent interest. It is closely related to [K, Theorem 2.1(b), p. 228] (which originates in [GK]), but our lemma has a more precise conclusion under a stronger hypothesis.

LEMMA 2.1. *If $A=[a_1, a_2, ..., a_n]$ is an $(n-1)$-by-n TP matrix, then, for $k=1, 2, ..., n$,*

$$a_k = \sum_{i=1,\, i\neq k}^{n} y_i a_i \tag{1}$$

in which $\operatorname{sgn}(y_i)$ *equals* $\operatorname{sgn}(-1)^i$ *if k is odd and* $\operatorname{sgn}(-1)^{i-1}$ *if k is even.*

Proof. If $k=1$, (1) has solution

$$y=[a_2, a_3, ..., a_n]^{-1} a_1$$

$$=\frac{1}{\det[a_2, a_3, ..., a_n]}\begin{bmatrix}\det[a_1, a_3, a_4, ..., a_n]\\ \det[a_2, a_1, a_4, ..., a_n]\\ \vdots\\ \det[a_2, a_3, ..., a_{n-1}, a_1]\end{bmatrix}$$

and $\operatorname{sgn}(y_i)=\operatorname{sgn}(-1)^i$. If $k>1$, then (1) has solution

$$y=[a_1, a_2, ..., a_{k-1}, a_{k+1}, ..., a_n]^{-1} a_k$$

$$=\frac{1}{\det[a_1, a_2, ..., a_{k-1}, a_{k+1}, ..., a_n]}$$

$$\times\begin{bmatrix}\det[a_k, a_2, a_3, ..., a_{k-1}, a_{k+1}, ..., a_n]\\ \det[a_1, a_k, a_3, ..., a_{k-1}, a_{k+1}, ..., a_n]\\ \vdots\\ \det[a_1, a_2, a_3, ..., a_{k-1}, a_{k+1}, ..., a_{n-1}, a_k]\end{bmatrix}$$

and we see that if k is odd, $\operatorname{sgn}(y_i)=\operatorname{sgn}(-1)^i$ while if k is even, $\operatorname{sgn}(y_i)=\operatorname{sgn}(-1)^{i-1}$. ∎

In other words, when expressing any column a_k of an $(n-1)$-by-n TP matrix A as a linear combination of the remaining columns, say $a_k=\sum_{i=1, i\neq k}^{n} y_i a_i$, all coefficients y_i are nonzero and, for $i=1, ..., k-1$ and $i=k+1, ..., n$, the signs of the coefficients alternate with y_{k-1} and y_{k+1} being positive. In terms of column insertion, this means the following: if we insert a column x into a square TP matrix A and remain TP, then, when x is expressed as a linear combination of the columns of A, the coefficients of the columns of A are "appropriately signed" in the following sense: the signs of the coefficients of the columns on each side of x alternate with the signs of the coefficients of the columns adjacent to x being positive. It is easy to construct examples to show that just inserting a positive column that is an appropriately signed linear combination of the columns of a square TP matrix A does not necessarily result in a TP matrix.

EXAMPLE 2.2. Consider the TP matrix

$$A=[a_1, a_2, a_3]=\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 5\\ 1 & 3 & 9\end{bmatrix}.$$

If we insert the column $x = a_1 + a_2 - \frac{1}{3} a_3$ between the first and second columns of A, we obtain

$$B = [a_1, x, a_2, a_3] = \begin{bmatrix} 1 & \frac{4}{3} & 1 & 2 \\ 1 & \frac{4}{3} & 2 & 5 \\ 1 & 1 & 3 & 9 \end{bmatrix}.$$

Since, for instance, the 2-by-2 minor in the lower left corner is negative, B is not TP.

Thus, the lemma is seen to be a necessary condition for inserting a row (column) in a TP matrix, but not a sufficient condition. In addition, care must be taken in choosing the relative magnitudes of the coefficients y_i so that a new TP matrix is created. The proof of our main result shows how to select these relative magnitudes so that we obtain a new TP matrix upon the insertion of the resulting column. Implicit in the proof is the use of Fekete's criterion [F] stated as follows: a matrix is totally positive if and only if the determinant of every square submatrix based on contiguous (e.g., $i, i+1, ..., i+k$) row and column index sets is positive.

THEOREM 2.3. *Let A be a TP matrix. Then, a line can be inserted between any pair of adjacent lines in A so that the resulting matrix is TP.*

Proof. Let A be a TP matrix. By transposition and/or external addition of rows/columns (see (iii) above), we may assume, without loss of generality, that A is square and of even order, say A is n-by-n in which $n = 2k$, and that we wish to insert a column in the middle. Specifically, let $A = [a_1, a_2, ..., a_k, a_{k+1}, ..., a_n]$ in which $n = 2k$ and let $\tilde{A} = [a_1, a_2, ..., a_k, x, a_{k+1}, ..., a_n]$ in which

$$x = \sum_{i=1}^{k} (-1)^{i+1} y_i a_{k-i+1} + \sum_{i=1}^{k} (-1)^{i+1} y_i a_{k+i}, \tag{2}$$

for some choice of $y_1, ..., y_k > 0$. Thus, the coefficients of x are appropriately signed and it remains to show that $y_1, ..., y_k > 0$ can be chosen so that $\tilde{A}$ is TP.

For each square contiguous submatrix $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x, let $\hat{a}_i$ denote the subcolumn of a_i having the same row indices as $\hat{x}$ ($i = 1, ..., n$), let $l(\hat{A})$ (respectively, $r(\hat{A})$) denote the number of columns of $\hat{A}$ which lie to the left (respectively, right) of $\hat{x}$, and let $m = m(\hat{A}) = \min\{l(\hat{A}), r(\hat{A})\}$. Thus,

$$\hat{A} = [\hat{a}_{k-l(\hat{A})+1}, ..., \hat{a}_k, \hat{x}, \hat{a}_{k+1}, ..., \hat{a}_{k+r(\hat{A})}] \tag{3}$$

in which

$$\hat{x} = \sum_{i=1}^{k} (-1)^{i+1} y_i \hat{a}_{k-i+1} + \sum_{i=1}^{k} (-1)^{i+1} y_i \hat{a}_{k+i}. \tag{4}$$

First, select $y_k > 0$. We now show that $y_k > 0$ ensures that all square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) = k-1$ have positive determinant or equivalently, that all square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) \geqslant k-1$ have positive determinant. (The latter statement follows since no square contiguous submatrix $\hat{A}$ of $\tilde{A}$ contains a subcolumn $\hat{x}$ of x and satisfies $m(\hat{A}) = k$.) We need to consider the following cases since $m(\hat{A}) = k-1$ implies that the order of $\hat{A}$ must be either n (Cases I or II) or $n-1$ (Case III).

Case I.

$$\begin{aligned}
\det \hat{A} &= \det[a_1, ..., a_k, x, a_{k+1}, ..., a_{n-1}] \\
&= \det[a_1, ..., a_k, (-1)^{k+1} y_k a_{2k}, a_{k+1}, ..., a_{2k-1}] \\
&= \det[a_1, ..., a_k, a_{k+1}, ..., a_{2k-1}, (-1)^{(k+1)+(k-1)} y_k a_{2k}] \\
&= y_k \det[a_1, ..., a_k, a_{k+1}, ..., a_{2k}] \\
&= y_k \det A > 0.
\end{aligned}$$

Case II. $\det \hat{A} = \det[a_2, ..., a_k, x, a_{k+1}, ..., a_n]$ is similar.

Case III.

$$\begin{aligned}
\det \hat{A} &= \det[\hat{a}_2, ..., \hat{a}_k, \hat{x}, \hat{a}_{k+1}, ..., \hat{a}_{n-1}] \\
&= \det[\hat{a}_2, ..., \hat{a}_k, (-1)^{k+1} y_k \hat{a}_1 + (-1)^{k+1} y_k \hat{a}_n, \hat{a}_{k+1}, ..., \hat{a}_{n-1}] \\
&= \det[(-1)^{(k+1)+(k-1)} y_k \hat{a}_1, \hat{a}_2, ..., \hat{a}_k, \hat{a}_{k+1}, ..., \hat{a}_{n-1}] \\
&\quad + \det[\hat{a}_2, ..., \hat{a}_k, \hat{a}_{k+1}, ..., \hat{a}_{n-1}, (-1)^{(k+1)+(k-1)} y_k \hat{a}_n] \\
&= y_k\{\det[\hat{a}_1, ..., \hat{a}_{n-1}] + \det[\hat{a}_2, ..., \hat{a}_n]\} \\
&> 0.
\end{aligned}$$

We will now show that $y_{k-1}, ..., y_1 > 0$ can be sequentially chosen so that $\hat{A}$ is TP. We will need the following observation.

Observation. Let $\hat{A}$ be given by (3) in which $m(\hat{A}) \geqslant j-1$ and $\hat{x}$ is given by (4). Then the terms of (4) involving any of $y_1, ..., y_{j-1}$ can be ignored when computing $\det \hat{A}$. (The terms of (4) involving any of $y_1, ..., y_{j-1}$

correspond to the columns $\hat{a}_{k-j+2}, ..., \hat{a}_k, \hat{a}_{k+1}, ..., \hat{a}_{k+j-1}$. Since $l(\hat{A})$, $r(\hat{A}) \geqslant j-1$, all of these are columns of $\hat{A}$ distinct from $\hat{x}$ and therefore can be ignored when computing det $\hat{A}$.)

For $j=k, k-1, ..., 2$, assume (inductively) that $y_k, y_{k-1}, ..., y_j$ have been chosen so that all square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) \geqslant j-1$ have positive determinant. Then we just need to show that there is $y_{j-1} > 0$ such that all square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) = j-2$ have positive determinant. (By the Observation, we will then have chosen $y_k, y_{k-1}, ..., y_j, y_{j-1} > 0$ such that all square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) \geqslant j-2$ have positive determinant.)

To simplify subscripting, let $m = j-2$ so that $y_k, ..., y_{m+2} > 0$ have been chosen such that all square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) \geqslant m+1$ have positive determinant and $y_{m+1} > 0$ needs to be chosen so that all square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) = m$ have positive determinant. We consider the various possibilities for $m = m(\hat{A})$ where $\hat{A}$ is given by (3) and $\hat{x}$ is given by (4). In each case (4) has a splitting $\hat{x} = s + t + u$ in which s is the sum of the terms in (4) each of whose vector part is a column of $\hat{A}$ distinct from $\hat{x}$ (and hence s can be ignored in computing det $\hat{A}$), t is the sum of the terms of $\hat{x} - s$ that involve y_{m+1}, and $u = \hat{x} - s - t$. Thus, the terms that sum to u involve the coefficients $y_k, ..., y_{m+2}$ only.

Case I. $m = r(\hat{A}) < l(\hat{A})$. Then $\hat{x} = s + t + u$ in which

$$s = \sum_{i=1}^{l(\hat{A})} (-1)^{i+1} y_i \hat{a}_{k-i+1} + \sum_{i=1}^{m} (-1)^{i+1} y_i \hat{a}_{k+i},$$

$$t = (-1)^{m+2} y_{m+1} \hat{a}_{k+m+1},$$

and

$$u = \sum_{i=1+l(\hat{A})}^{k} (-1)^{i+1} y_i \hat{a}_{k-i+1} + \sum_{i=m+2}^{k} (-1)^{i+1} y_i \hat{a}_{k+i}.$$

Thus,

$$\begin{aligned}\det \hat{A} &= \det[\hat{a}_{k-l(\hat{A})+1}, ..., \hat{a}_k, t+u, \hat{a}_{k+1}, ..., \hat{a}_{k+m}] \\ &= \det[\hat{a}_{k-l(\hat{A})+1}, ..., \hat{a}_k, \hat{a}_{k+1}, ..., \hat{a}_{k+m}, (-1)^{2m+2} y_{m+1} \hat{a}_{k+m+1}] \\ &\quad - \det[\hat{a}_{k-l(\hat{A})+1}, ..., \hat{a}_k, -u, \hat{a}_{k+1}, ..., \hat{a}_{k+m}] \\ &> 0,\end{aligned}$$

or equivalently,

$$y_{m+1} > b(\hat{A}) = \frac{\det[\hat{a}_{k-l(\hat{A})+1}, \ldots, \hat{a}_k, -u, \hat{a}_{k+1}, \ldots, \hat{a}_{k+m}]}{\det[\hat{a}_{k-l(\hat{A})+1}, \ldots, \hat{a}_k, \hat{a}_{k+1}, \ldots, \hat{a}_{k+m}, \hat{a}_{k+m+1}]}. \qquad \text{(i)}$$

Case II. $m = r(\hat{A}) = l(\hat{A})$. Then $\hat{x} = s + t + u$ in which

$$s = \sum_{i=1}^{m} (-1)^{i+1} y_i \hat{a}_{k-i+1} + \sum_{i=1}^{m} (-1)^{i+1} y_i \hat{a}_{k+i},$$

$$t = (-1)^{m+2} y_{m+1} \hat{a}_{k-m} + (-1)^{m+2} y_{m+1} \hat{a}_{k+m+1},$$

and

$$u = \sum_{i=m+2}^{k} (-1)^{i+1} y_i \hat{a}_{k-i+1} + \sum_{i=m+2}^{k} (-1)^{i+1} y_i \hat{a}_{k+i}.$$

Thus,

$$\begin{aligned}
\det \hat{A} &= \det[\hat{a}_{k-m+1}, \ldots, \hat{a}_k, t+u, \hat{a}_{k+1}, \ldots, \hat{a}_{k+m}] \\
&= \det[(-1)^{2m+2} y_{m+1} \hat{a}_{k-m}, \hat{a}_{k-m+1}, \ldots, \hat{a}_k, \hat{a}_{k+1}, \ldots, \hat{a}_{k+m}] \\
&\quad + \det[\hat{a}_{k-m+1}, \ldots, \hat{a}_k, \hat{a}_{k+1}, \ldots, \hat{a}_{k+m}, (-1)^{2m+2} y_{m+1} \hat{a}_{k+m+1}] \\
&\quad - \det[\hat{a}_{k-m+1}, \ldots, \hat{a}_k, -u, \hat{a}_{k+1}, \ldots, \hat{a}_{k+m}] \\
&> 0,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
y_{m+1} &> b(\hat{A}) \\
&= \frac{\det[\hat{a}_{k-m+1}, \ldots, \hat{a}_k, -u, \hat{a}_{k+1}, \ldots, \hat{a}_{k+m}]}{\det[\hat{a}_{k-m}, \ldots, \hat{a}_{k+m}] + \det[\hat{a}_{k-m+1}, \ldots, \hat{a}_{k+m+1}]}.
\end{aligned} \qquad \text{(ii)}$$

Case III. $m = l(\hat{A}) < r(\hat{A})$. Analogously to Case I, one has that

$$y_{m+1} > b(\hat{A}) = \frac{\det[\hat{a}_{k-m+1}, \ldots, \hat{a}_k, -u, \hat{a}_{k+1}, \ldots, \hat{a}_{k+r(\hat{A})}]}{\det[\hat{a}_{k-m}, \ldots, \hat{a}_k, \hat{a}_{k+1}, \ldots, \hat{a}_{k+r(\hat{A})}]}. \qquad \text{(iii)}$$

Notice that in each of the Cases I, II, and III, $y_{m+1} > 0$ can be determined from $y_k, \ldots, y_{m+2}$ so that $\det \hat{A} > 0$. Since there are a finite number of square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) = m$, we can choose $y_{m+1} > 0$ large enough so that for *all* square contiguous submatrices $\hat{A}$ of $\tilde{A}$ containing a subcolumn $\hat{x}$ of x and satisfying $m(\hat{A}) = m$, $\det \hat{A} > 0$. This completes the proof. ∎

The proof of the theorem yields the following algorithm for row/column insertion of TP matrices.

(COLUMN) INSERTION ALGORITHM. Suppose that A is a p-by-q TP matrix and we wish to insert a column between the jth and $(j+1)$st column of A so that the resulting matrix is TP.

Step 1. Convert A to a square TP matrix B of even order $n=2k$ and with the jth and $(j+1)$st columns of A lying in the middle of B (by external addition of rows/columns).

Step 2. Set $y_1, ..., y_k = 1$ and $i = 0$.

Step 3. Convert B to $\tilde{A} = [a_1, a_2, ..., a_k, x, a_{k+1}, ..., a_n]$ by inserting the column x defined by (2) in the middle of B.

Step 4. $i = i+1$.

Step 5. $\text{glb}(k-i) = \max\{b(\hat{A}) :$ $\hat{A}$ is a square contiguous submatrix of $\tilde{A}$ having $k+1$ as a column index and satisfying $m(\hat{A}) = k-i-1$ and $b(\hat{A})$ is as defined in the proof of the Theorem$\}$.

Step 6. $y_{k-i} = \max\{1, \text{glb}(k-i) + 0.0001\}$.

Step 7. If $i \neq k-1$, return to Step 4.

Step 8. Set $x = \sum_{i=1}^{k} (-1)^{i+1} y_i a_{k-i+1} + \sum_{i=1}^{k} (-1)^{i+1} y_i a_{k+i}$.

Step 9. Delete each row/column of $\tilde{A}$ that corresponds to a row/column added externally to A. The resulting matrix is the desired TP matrix.

Closing Remark. Note that to do a row insertion instead, we simply perform the algorithm on the transpose of our original matrix.

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